THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010C/D Advanced Calculus 2019-2020

Solution to Midterm Examination

1. (*12 pts*) Answer the following questions.

   (a) Find the equation of plane Π passing through the point $(2,3,-1)$ and parallel to the plane $3x-4y+7z=1$.

   (b) Find the distance between the two planes in part (a).

   (c) Find the angle between the plane Π and the plane $8x+3y-z=2$.

Ans:

(a) Since the plane Π is parallel to $3x-4y+7z=1$, it has an equation of the form $3x-4y+7z=a$ for some real number a.

Put $(2,3,-1)$ into it, we have $3(2)-4(3)+7(-1)=a$ and so $a=-13$.

Therefore, equation of Π: $3x-4y+7z=-13$.

(b) **Method 1:**

By formula, distance $= \left|\dfrac{1-(-13)}{\sqrt{(3)^2+(-4)^2+7^2}}\right| = \dfrac{14}{\sqrt{74}} = \dfrac{7\sqrt{74}}{37}$.

Method 2:

Let L be the line through $(2,3,-1)$, since $L\perp\Pi$, L can be given by the following parametric equation:

$$L(t)=(2,3,-1)+t(3,-4,7)=(2+3t,3-4t,-1+7t)$$

Put it into $3x-4y+7z=1$, we have

$$\begin{aligned}3(2+3t)-4(3-4t)+7(-1+7t) &= 1\\ -13+74t &= 1\\ t &= \frac{7}{37}\end{aligned}$$

Therefore, L intersects the plane $3x-4y+7z=1$ at $L(\frac{7}{37})$ and the distance between the planes

$$=\left\|L(\frac{7}{37})-L(0)\right\| = \left\|\frac{7}{37}(3,-4,7)\right\| = \frac{7}{37}\sqrt{3^2+(-4)^2+7^2} = \frac{7\sqrt{74}}{37}$$

.

(c)

$$\begin{aligned}\text{Angle between planes} &= \text{Angle between normals}\\ &= \cos^{-1}\left(\frac{(8,3,-1)\cdot(3,-4,7)}{\|(8,3,-1)\|\|(3,-4,7)\|}\right)\\ &= \cos^{-1}\frac{5}{74}\end{aligned}$$

2. (*6 pts*) Compute the arclength of the curve $\gamma(t)=(t^2,2t,\ln t)$ for $1\le t\le 5$.

Ans:

We have $\gamma(t) = (t^2, 2t, \ln t)$ and $\gamma'(t) = (2t, 2, \frac{1}{t})$. Then,

$$\begin{aligned}
\text{Arclength} &= \int_1^5 \|\gamma'(t)\|\, dt \\
&= \int_1^5 \sqrt{(2t)^2 + 2^2 + (\frac{1}{t})^2}\, dt \\
&= \int_1^5 \sqrt{4t^2 + 4 + \frac{1}{t^2}}\, dt \\
&= \int_1^5 \sqrt{(2t + \frac{1}{t})^2}\, dt \\
&= \int_1^5 2t + \frac{1}{t}\, dt \\
&= [t^2 + \ln t]_1^5 \\
&= 24 + \ln 5
\end{aligned}$$

3. (*10 pts*) Evaluate the following limits or show they do not exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y - y^3}{x^2 + y^2}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y - x^2y - 3xy^2}{x^4 + y^2}$

Ans:

(a)

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2y - y^3}{x^2 + y^2} &= \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta - r^3\sin^3\theta}{r^2} \\
&= \lim_{r\to 0} r(\cos^2\theta\sin\theta - \sin^3\theta) \\
&= 0 \qquad \text{(By sandwich theorem)}
\end{aligned}$$

(b) We study the limits along different paths.

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^3y - x^2y - 3xy^2}{x^4 + y^2} = \lim_{y\to 0} \frac{(0)^3y - (0)^2y - 3(0)y^2}{(0)^4 + y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} \frac{x^3(x^2) - x^2(x^2) - 3x(x^2)^2}{x^4 + (x^2)^2} = \lim_{x\to 0} \frac{-2x^5 - x^4}{2x^4} = \lim_{x\to 0} \frac{-2x - 1}{2} = -\frac{1}{2}$$

The two paths give different limits and so $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y - x^2y - 3xy^2}{x^4 + y^2}$ does not exist.

4. (*10 pts*) Let $f(x,y) = \dfrac{e^{xy+6}}{1 + 4x + 3y}$.

   (a) Find df, the differential of f.

   (b) Use the result of (a) to approximate the change in f when (x,y) changes from $(-2,3)$ to $(-1.9, 2.95)$.

Ans:

(a) $f(x,y) = \dfrac{e^{xy+6}}{1 + 4x + 3y}$ and so

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{(4x + 3y + 1)ye^{xy+6} - 4e^{xy+6}}{(4x + 3y + 1)^2}dx + \frac{(4x + 3y + 1)xe^{xy+6} - 3e^{xy+6}}{(4x + 3y + 1)^2}dy.$$

(b) Put $(x, y) = (-2, 3)$, $dx = -1.9 - (-2) = 0.1$ and $dy = 2.95 - 3 = -0.05$, so

$$\Delta f \approx df = \frac{(2)(3)e^0 - (4)e^0}{(2)^2}(0.1) + \frac{(2)(-2)e^0 - (3)e^0}{(2)^2}(-0.05) = 0.05 + 0.0875 = 0.1375$$

5. (*10 pts*) Lef $f(x, y) = \ln(30 - 10x + x^2 + y^2)$.

(a) Draw the level set of f through the point $(2, 4)$. Label all its intercept(s).

(b) Find the direction where f decreases most rapidly at the point $(2, 4)$.

Ans:

(a) Lef $f(x, y) = \ln(30 - 10x + x^2 + y^2)$ and then $f(2, 4) = \ln 30$.
If $f(x, y) = f(2, 4)$, then $\ln(30 - 10x + x^2 + y^2) = \ln 30$ and so $x^2 - 10x + y^2 = 0$. It can be expressed as $(x - 5)^2 + y^2 = 5^2$ which gives the circle centered at $(5, 0)$ with radius 5.

(b) Note that $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (\frac{-10 + 2x}{30 - 10x + x^2 + y^2}, \frac{2y}{30 - 10x + x^2 + y^2})$.
Therefore, the direction where f decreases most rapidly at the point $(2, 4) = -\nabla f(2, 4) = (\frac{1}{5}, -\frac{4}{15})$.

6. (*10 pts*) Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is C^∞ function and there exists a positive integer n such that $f(tx, ty, tz) = t^n f(x, y, z)$ for all $t \in \mathbb{R}$ and $(x, y, z) \in \mathbb{R}^3$.

(a) Show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf.$$

(b) Suppose $n = 7$ and $\frac{\partial f}{\partial x \partial y \partial z}(1, 1, 1) = 1$. Find the value of $\frac{\partial f}{\partial x \partial y \partial z}(-3, -3, -3)$.

Ans:

(a) We have $f(tx, ty, tz) = t^n f(x, y, z)$, then we differentiate both sides with respect to t and get

$$x\frac{\partial f}{\partial x}(tx, ty, tz) + y\frac{\partial f}{\partial y}(tx, ty, tz) + z\frac{\partial f}{\partial z}(tx, ty, tz) = nt^{n-1} f(x, y, z).$$

Put $t = 1$, then we have

$$f(tx, ty, tz) = t^n f(x, y, z)$$

(b)

$$\begin{aligned}
& & f(tx, ty, tz) &= t^n f(x, y, z) \\
&(\text{Take } \frac{\partial}{\partial z}) & t\frac{\partial f}{\partial z}(tx, ty, tz) &= t^n \frac{\partial f}{\partial z}(x, y, z) \\
&(\text{Take } \frac{\partial}{\partial y}) & t^2\frac{\partial^2 f}{\partial y \partial z}(tx, ty, tz) &= t^n \frac{\partial^2 f}{\partial y \partial z}(x, y, z) \\
&(\text{Take } \frac{\partial}{\partial x}) & t^3\frac{\partial^3 f}{\partial x \partial y \partial z}(tx, ty, tz) &= t^n \frac{\partial^2 f}{\partial x \partial y \partial z}(x, y, z)
\end{aligned}$$

Put $t = -3$, $n = 7$, $x = y = z = 1$, we have

$$\begin{aligned}
(-3)^3\frac{\partial^3 f}{\partial x \partial y \partial z}(-3, -3, -3) &= (-3)^7\frac{\partial^3 f}{\partial x \partial y \partial z}(1, 1, 1) \\
\frac{\partial^3 f}{\partial x \partial y \partial z}(-3, -3, -3) &= (-3)^4(1) \\
&= 81
\end{aligned}$$

7. (*22 pts*) Let

$$f(x,y) = \begin{cases} \sqrt[3]{xy^2}\sin\dfrac{x}{y} & \text{if} \quad y \neq 0; \\ \\ 0 & \text{if} \quad y = 0. \end{cases}$$

(a) Show that f is continuous at $(0,0)$.

(b) Show that $\dfrac{\partial f}{\partial x}(0,0) = 0$ and $\dfrac{\partial f}{\partial y}(0,0) = 0$.

(c) Let $\mathbf{u} = \left(-\dfrac{3}{5}, \dfrac{4}{5}\right)$. Compute the directional derivative $\nabla_{\mathbf{u}} f(0,0) = D_{\mathbf{u}} f(0,0)$.

(d) Determine all the point(s) for which f is differentiable? Prove your assertion.

Ans:

(a) Note that $0 \leq |f(x,y)| \leq |\sqrt[3]{xy^2}|$ near $(0,0)$ and $\lim\limits_{(x,y)\to(0,0)} |\sqrt[3]{xy^2}| = \lim\limits_{r\to 0} r|\sqrt[3]{\cos\theta\sin^2\theta}| = 0$.

By sandwich theorem, $\lim\limits_{(x,y)\to(0,0)} |f(x,y)| = 0$ which implies $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

We have $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ and so f is continuous at $(0,0)$.

Comment: When you evaluate $\lim\limits_{(x,y)\to(0,0)} f(x,y)$, you are looking at the behaviour of the function $f(x,y)$ near the point $(0,0)$, so you may not say $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \sqrt[3]{xy^2}\sin\dfrac{x}{y}$ since $f(x,y) = 0$ but not $\sqrt[3]{xy^2}\sin\dfrac{x}{y}$ when $(x,y) = (x,0)$.

(b) We have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt[3]{(0)(k)^2}\sin(\frac{0}{k})-0}{h} = 0.$$

(c)

$$\begin{aligned} D_{\mathbf{u}} f(0,0) &= \lim_{t\to 0}\frac{f(t\mathbf{u})-f(\mathbf{0})}{t} \\ &= \lim_{t\to 0}\frac{\sqrt[3]{(-\frac{3}{5}t)(\frac{4}{5}t)^2}\sin\left(\frac{-\frac{3}{5}t}{\frac{4}{5}t}\right)}{t} \\ &= \lim_{t\to 0}\sqrt[3]{-\frac{48}{125}}\sin(-\frac{3}{4}) \end{aligned}$$

Comment: We have the fact that if f is differentiable at $(0,0)$, then $\nabla f(0,0)\cdot\mathbf{u} = D_{\mathbf{u}} f(0,0)$. However, we do not know whether f is differentiable at $(0,0)$ at this moment (and in fact it is not).

(d)
- Note that $\nabla f(0,0)\cdot\mathbf{u} = \mathbf{0}\cdot\mathbf{u} = 0 \neq D_{\mathbf{u}} f(0,0)$, so f is not differentiable at $(0,0)$.
- For $x \neq 0$,

$$\begin{aligned} \frac{\partial f}{\partial y}(x,0) &= \lim_{k\to 0}\frac{f(x,k)-f(x,0)}{k} \\ &= \lim_{k\to 0}\frac{\sqrt[3]{xk^2}\sin(\frac{x}{k})}{k} \\ &= \lim_{k\to 0}\sqrt[3]{\frac{x}{k}}\sin(\frac{x}{k}) \end{aligned}$$

which does not exist for any $x \neq 0$.

Therefore, f is not differentiable at $(x,0)$ for $x \neq 0$.

- For $y \neq 0$, $f(0,y) = 0$ and so $\dfrac{\partial f}{\partial y}(0,y) = 0$. Also,

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,y) &= \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt[3]{hy^2}\sin(\frac{h}{y}) - 0}{h} \\
&= \lim_{h\to 0} (\frac{y}{h})^{2/3} \sin(\frac{h}{y}) \\
&= \lim_{h\to 0} (\frac{h}{y})^{2/3} \left[\frac{\sin(\frac{h}{y})}{(\frac{h}{y})}\right] \\
&= (0)(1) \\
&= 0
\end{aligned}$$

For $x \neq 0$, $y \neq 0$,

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= \frac{1}{3}(\frac{x}{y})^{-2/3}\sin(\frac{x}{y}) + (\frac{x}{y})^{1/3}\cos(\frac{x}{y}) \\
\frac{\partial f}{\partial y}(x,y) &= \frac{2}{3}(\frac{x}{y})^{1/3}\sin(\frac{x}{y}) - (\frac{x}{y})^{1/3}(\frac{1}{y})\cos(\frac{x}{y})
\end{aligned}$$

Both $\dfrac{\partial f}{\partial x}(x,y)$ and $\dfrac{\partial f}{\partial y}(x,y)$ are continuous and $\displaystyle\lim_{(x,y)\to(0,y_0)} \frac{\partial f}{\partial x}(x,y) = \lim_{(x,y)\to(0,y_0)} \frac{\partial f}{\partial y}(x,y) = 0$ for $y_0 \neq 0$. Hence, f is C^1 on $\{(x,y) \in \mathbb{R}^2 : y \neq 0\}$ which implies $f(x,y)$ is differentiable for $y \neq 0$.

Therefore, the set where f is differentiable is $\{(x,y) \in \mathbb{R}^2 : y \neq 0\}$.

Comment: Since the function $\sqrt[3]{x}$ is not differentiable at $x = 0$, you may not say $\sqrt[3]{xy^2}$ is differentiable for all $(x,y) \in \mathbb{R}^2$.